

STUDY OF A NEW VARIANT OF PAIRWISE PARACOMPACTNESS ON BITOPOLOGICAL SPACES

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ABSTRACT. The present paper deals with a new variant of pairwise paracompactness with the help of a new kind of closure operator. This closure operator is named as $\Omega_{(i,j)}$ operator and is defined over a bitopological space [15] whereas this new variant of pairwise paracompactness has been named as pairwise Ω -paracompactness. The concept of pairwise Ω -paracompactness deals with the countable collection of open subsets of a given set. It is observed that the concept of pairwise Ω -paracompactness happens to be a more relaxed condition than a finite collection of open sets and hence encompasses a greater domain of bitopological spaces. Further various characterizations of pairwise Ω -paracompactness are studied and the authors explore it in the presence of existing separation axioms too such as pw antilocally countable, pw T_1 , pw T_2 , pw T_3 and pw p-space etc. which leads to the emergence of a number of properties of pairwise Ω -paracompactness in the presence of these separation axioms.

2000 MATHEMATICS SUBJECT CLASSIFICATION. 54A05, 54D10, 54D15, 54D20, 54E55

KEYWORDS AND PHRASES. Pairwise $\Omega_{(i,j)}$ open sets, $\mathcal{R}_1$ Topology, $\mathcal{R}_2$ Topology, pairwise Ω -paracompactness.

SUBMISSION DATE. 15.10.2025

1. INTRODUCTION

The concept of locally finite coverings has been an interesting feature of topological and bitopological developments and has played a vital role in the emergence of various concepts such as paracompactness, compactness, Lindelöf space etc. The notion of paracompactness was established in terms of locally finite coverings as an extension of the concept of compactness [10, 17]. In 1963, Kelly noticed that the asymmetric behavior of a quasi-metric space gave rise to two topologies on it and with this realization, he established the concept of Bitopological Spaces [7] where any set X is governed by two topologies say $\mathcal{T}_1$ and $\mathcal{T}_2$ and is represented as $(X, \mathcal{T}_1, \mathcal{T}_2)$. With the emergence of the concept of Bitopological Spaces, various concepts developed on topological spaces have been carried forward on bitopological spaces and their nature and behaviour have been studied extensively. Paracompactness in the bitopological setting appears to be one of the most intractable of these concepts. The concept of paracompactness has extensively been studied by a number of mathematicians from time to time and various versions of paracompactness have been produced in the field of both topological as well as bitopological spaces. Fuad and Hasan [4] coined the concept of p-[a, b] paracompactness and studied it on bitopological spaces. Harjot S.

[5] analysed the concept of pairwise basis for bitopological spaces whereas Kalantan [14] studied L-paracompactness and L2-paracompactness. Further it was only natural to derive new topologies on a topological space by imposing various conditions on given topological spaces and hence a plethora of derived bitopological spaces found their existence. Following the lineage, Ω topology was introduced on a topological space [3] and further two new topologies $\mathcal{R}_1$ topology and $\mathcal{R}_2$ topology were introduced on a Bitopological Space [15]. These new topologies paved way for a new Bitopological Space $(X, \mathcal{R}_1, \mathcal{R}_2)$ which is a derived bitopology on a given Bitopological Space. In the present work, we initiate the concept named as pairwise Ω -paracompactness on a bitopological space by utilizing $\mathcal{R}_1$ locally finite covers and $\mathcal{R}_2$ locally finite covers. We provide several characterizations of this pairwise Ω -paracompactness and study the conditions under which a relation between pairwise Ω -paracompactness and pairwise paracompactness can be established.

2. PRELIMINARIES

Throughout the paper, to represent a bitopological space or BTS, let us use the notation $(X, \mathcal{T}_1, \mathcal{T}_2)$ with X as a non-empty set possessing topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. The concepts of interior and closure of any subset of the set maintain their standard meanings. To quote pairwise concepts, we use the notation pw. Further, to make this research work efficient enough, let us recollect a few well-known definitions. The definition of pw T_1 space was given by Kelly [7] as follows:

Definition 2.1. Any BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pw T_1 provided for any pair of distinct elements of X say p and q , there exists an open set in $\mathcal{T}_i$ say P and another open set in $\mathcal{T}_j$ say Q such that P possesses p but not q whereas Q possesses q but not p for all $i, j \in \{1, 2\}$, $i \neq j$.

Kelly [7] further extended the field of separation axioms over the field of BTS with the introduction of some more axioms such as:

Definition 2.2. Any BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pw T_2 space or pw Hausdorff space if for each pair of distinct elements of X say p and q , there exists a pair of mutually disjoint open sets say P in $\mathcal{T}_i$ and Q in $\mathcal{T}_j$ such that P possesses p but not q and Q possesses q but not p for all $i \in \{1, 2\}$, $i \neq j$.

Definition 2.3. Any BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pw regular space whenever for each element p of X and each $\mathcal{T}_i$ open set P of X possessing p , it is possible to find a $\mathcal{T}_i$ open set Q possessing p whose $\mathcal{T}_j$ closure is contained in P for all $i \in \{1, 2\}$, $i \neq j$.

Definition 2.4. Any BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is defined as pw T_3 whenever it is both pw regular and pw T_1 space.

Definition 2.5. Any BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is defined as pw normal space whenever for every $\mathcal{T}_i$ closed set S of X and every $\mathcal{T}_j$ open set P of X containing S , it is always possible to find another $\mathcal{T}_j$ open set Q possessing S whose $\mathcal{T}_i$ closure is within P for all $i \in \{1, 2\}$, $i \neq j$.

In addition to this, further concepts were introduced from time to time:

Definition 2.6. Any BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is defined as **pw T_4** if it is both pw normal and pw T_1 space.

Definition 2.7. Any BTS is **pw p-space** if the union of countably many $\mathcal{T}_i$ closed sets is $\mathcal{T}_i$ closed for all $i \in \{1, 2\}$.

Definition 2.8. Any topological space $(X, \mathcal{T})$ is defined as **paracompact** whenever each open cover $\mathcal{U}$ in X receives a refinement which is both locally finite as well as open. By open refinement, we mean a collection of open sets $\mathcal{V}$ in X whose elements are contained in some element of $\mathcal{U}$ and by locally finite, we mean that each element in X has a neighborhood that intersects finitely many elements of $\mathcal{V}$.

The concept of paracompactness further paved the way for the concept of pw paracompactness. The idea of pw paracompactness was presented by Fletcher et al. with a certain limitation that whenever the BTS under consideration happens to be pw paracompact and pw Hausdorff [17], then both the topologies coincide and the BTS becomes a single topological space. Later Datta [10] and Raghavan and Reilly [18] defined the notion of pw paracompactness in different ways. Raghavan and Reilly [18] gave an interesting notion to define a variety of pw paracompactness on a BTS as follows:

Definition 2.9. Any BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is (τ, σ) paracompact with respect to ρ if each τ open cover admits of a σ open refinement which happens to be ρ locally finite. We give an equivalent setting to this arrangement as X is (τ, σ, ρ) paracompact where τ , σ and ρ are the topologies defined over X . With the different choices of τ , σ and ρ , various notions of paracompactness can be defined. For instance, $\tau = \sigma \neq \rho$ defines RR-pw paracompactness whereas $\tau \neq \sigma = \rho$ defines FHP paracompactness.

3. PW Ω -PARACOMPACTNESS

The concept of Ω -topology is a derived topology in a given topological space and it is defined in terms of Ω -open sets [3] as follows:

Definition 3.1. In a topological space $(X, \mathcal{T})$, a subset $A \subseteq X$ is defined to be Ω -open if $\forall x \in A$ there is an open set U containing x such that $cl(U) - A$ is countable.

This concept of Ω -topology can further be taken on BTSs [15] which leads to the introduction of a new topology on a given BTS in the following manner:

Definition 3.2. (a) In a BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$, a subset $U \subseteq X$ is defined to be $\Omega_{(1,2)}$ open if each element x of U there exists a $\mathcal{T}_2$ open neighbourhood U_x of x such that $(\mathcal{T}_1 cl(U_x)) - U$ is countable.

(b) In a BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ a subset $A \subseteq X$ is defined to be $\Omega_{(2,1)}$ open if each element x of A has a $\mathcal{T}_1$ open neighborhood U_x of x such that $(\mathcal{T}_2 cl(U_x)) - A$ is countable.

For the lucidity of the concept, we present the evidence for $\Omega_{(1,2)}$ operator to define a topology on $(X, \mathcal{T}_1, \mathcal{T}_2)$.

- (i) It is obvious that $\emptyset$ and X are $\Omega_{(1,2)}$ open.

- (ii) Let $\mathcal{U} = \{A_\alpha | \alpha \in \Delta\}$ be a collection of $\Omega_{(1,2)}$ open sets. Let $x \in \bigcup_{\alpha \in \Delta} A_\alpha$, then $x \in A_\alpha$ for some $\alpha \Rightarrow$ there exists a $\mathcal{T}_2$ open set U_2 such that $x \in U_2$ and $\mathcal{T}_1 \text{cl}(U_2) - A_\alpha$ is countable. Hence, $\bigcup_{\alpha \in \Delta} (\mathcal{T}_1 \text{cl}(U_2) - U_\alpha) = \bigcup_{\alpha \in \Delta} A_\alpha$ is countable.
- (iii) Let A_1 and A_2 be $\Omega_{(1,2)}$ open sets. Let $x \in A_1 \cap A_2, \Rightarrow$ there exist $\mathcal{T}_2$ open sets U_1 and U_2 such that $x \in U_1 \cap U_2$ and both $\mathcal{T}_1 \text{cl}(U_1) - A_1$ and $\mathcal{T}_1 \text{cl}(U_2) - A_2$ are countable. Consider $U = U_1 \cap U_2, \Rightarrow \mathcal{T}_1 \text{cl}(U) - A_1 \cap A_2 = (\mathcal{T}_1 \text{cl}(U) - A_1) \cup (\mathcal{T}_1 \text{cl}(U) - A_2)$ is countable.

With the help of $\Omega_{(1,2)}$ and $\Omega_{(2,1)}$ operators, two new topologies can be defined on the BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$. For the sake of convenience, we denote the topology defined in terms of $\Omega_{(1,2)}$ open sets as $\mathcal{R}_1$ topology and the topology defined in terms of $\Omega_{(2,1)}$ open sets as $\mathcal{R}_2$ topology. Consequently, it is always possible to derive a new BTS $(X, \mathcal{R}_1, \mathcal{R}_2)$ from the parent $\text{BTS}(X, \mathcal{T}_1, \mathcal{T}_2)$. With the help of these new topologies, we now introduce a pair of paracompactness on the BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ as follows:

Definition 3.3. A BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *pw Ω -paracompact* if it is both $(\mathcal{T}_1, \mathcal{T}_1)$ paracompact with respect to $\mathcal{R}_1$ and $(\mathcal{T}_2, \mathcal{T}_2)$ paracompact with respect to $\mathcal{R}_2$.

Definition 3.4. Any BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is $(\mathcal{T}_1, \mathcal{T}_1)$ paracompact with respect to $\mathcal{R}_1$ if every $\mathcal{T}_1$ open cover is always equipped with a $\mathcal{T}_1$ open refinement which is $\mathcal{R}_1$ locally finite.

Example 3.5. Let us consider two topologies on the set of real numbers R as follows:

$$(R, \mathcal{T}_1) = \{\emptyset, R, \{0\}, (-\infty, 0), (-\infty, 0]\}$$

$$(R, \mathcal{T}_2) = \{\emptyset, R, \{0\}, (0, \infty), [0, \infty)\}$$

$(R, \mathcal{R}_1)$ consists of all the sets of the type $\emptyset, [0, \infty) - C, R - C$ whereas $(R, \mathcal{R}_2)$ consists of all the sets of the type $\emptyset, (-\infty, 0] - C, R - C$ where C is any countable subset of R . Further, every $\mathcal{T}_1$ open cover is itself $\mathcal{T}_1$ open refinement which is $\mathcal{R}_1$ locally finite as an $\mathcal{R}_1$ neighbourhood of any element x of R can intersect only finitely many elements of $\mathcal{T}_1$ open refinement. Similarly, every $\mathcal{T}_2$ open cover is itself $\mathcal{T}_2$ open refinement which is $\mathcal{R}_2$ locally finite as an $\mathcal{R}_2$ neighbourhood of any element x of R can intersect only finitely many elements of $\mathcal{T}_2$ open refinement. Hence the space is **pw Ω -paracompact**.

Example 3.6. Let R be the set of real numbers and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the usual topology and right order topology respectively. Then, $\mathcal{R}_1$ is the topology finer than $\mathcal{T}_2$ as it consists of all the members of it along with all the elements of the type $A - C$ and $R - C$ where A is any member of $\mathcal{T}_2$. Members of $\mathcal{R}_2$ will be of the type $\emptyset, (-\infty, a) - C$ and $R - C$ for each element a of R . This space is not **pw Ω -paracompact**.

Theorem 3.7. A collection of subsets of X is $\mathcal{R}_1$ -locally finite if and only if it is $\mathcal{R}_1$ -closure preserving.

Proof: Consider $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ as a collection of subsets of X which happens to be $\mathcal{R}_1$ -locally finite. Then, $\bigcup (\mathcal{R}_1\text{-cl}(U_\alpha)) \subseteq (\mathcal{R}_1\text{-cl}(\bigcup U_\alpha))$ is obvious. Conversely, let $x \notin \bigcup (\mathcal{R}_1\text{-cl}(U_\alpha))$ implies $x \notin (\mathcal{R}_1\text{-cl}(U_\alpha))$ for all $\alpha \in \Delta$. Now $\mathcal{U}$ being $\mathcal{R}_1$ -locally finite implies there exists $U_x \in \mathcal{R}_1$

containing x in such a way that $U_x \cap U_\alpha = \emptyset$ except for finitely many α say $\alpha = 1, 2, 3, \dots, n$. For these α we have $V_\alpha \in \mathcal{R}_1$ containing x which are disjoint from each of U_α for $\alpha = 1, 2, 3, \dots, n$. Consider $V = U_x \cap V_1 \cap V_2 \dots \cap V_n$. Then V is $\mathcal{R}_1$ -open and is disjoint from each of U_α .

Corollary 3.8. *An $\mathcal{R}_1$ locally finite collection of subsets of X within a BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is $\mathcal{T}_1$ closure preserving if and only if it is $\mathcal{T}_1$ finer than countable topology.*

Proof: Similar to Theorem 3.7.

Theorem 3.9. *In a pw regular BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$, $\mathcal{T}_1 \subseteq \mathcal{R}_2$ and $\mathcal{T}_2 \subseteq \mathcal{R}_1$.*

Proof: Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a pw regular BTS. Let $x \in U_x \in \mathcal{T}_1$. Then due to pw regularity of X , there exists $x \in V_x \in \mathcal{T}_1$ containing x such that $\mathcal{T}_2\text{-cl}(V_x) \subseteq U_x \Rightarrow U_x \in \mathcal{R}_2$.

Corollary 3.10. *A $\mathcal{T}_1$ locally finite collection of subsets of the set X where $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a pw regular BTS is $\mathcal{R}_2$ locally finite.*

Theorem 3.11. *If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a regular BTS and if $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ is an $\mathcal{R}_1$ -locally finite collection of subsets of X then $\mathcal{U}' = \{\mathcal{R}_1\text{-cl}(U_\alpha) | \alpha \in \Delta\}$ is also $\mathcal{R}_1$ -locally finite.*

Proof Consider $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ as an $\mathcal{R}_1$ -locally finite collection of subsets taken from the set X . Take an arbitrary element $x \in X$ then there always exists $U_x \in \mathcal{R}_1$ containing x in such a way that $U_x \cap U_\alpha = \emptyset$ except for finitely many values of α . Now $U_x \in \mathcal{R}_1 \Rightarrow U_x \cap \mathcal{R}_1\text{-cl}(U_\alpha) = \emptyset$ except for finitely many α . Hence $\mathcal{U}' = \{\mathcal{R}_1\text{-cl}(U_\alpha) | \alpha \in \Delta\}$ is also $\mathcal{R}_1$ -locally finite.

A BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is defined as $(\mathcal{T}_1, \mathcal{T}_1)$ paracompact with respect to $\mathcal{T}_2$ if every $\mathcal{T}_1$ open cover has a $\mathcal{T}_1$ open refinement which is $\mathcal{T}_2$ locally finite. But a $\mathcal{T}_2$ locally finite collection of subsets of the set X in a pw regular BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is $\mathcal{R}_2$ locally finite. This leads to an important conclusion that in a pw regular BTS, pw paracompactness implies pw Ω -paracompactness.

Theorem 3.12. *If U is a $\mathcal{T}_1$ -open set in a BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$, then $\mathcal{R}_1\text{-cl}(U) \subseteq \mathcal{T}_2\text{-cl}(U)$.*

Proof. Let $x \notin \mathcal{T}_2\text{-cl}\{U\} \Rightarrow \exists$ a $\mathcal{T}_2$ neighborhood U_x of x in X disjoint from U . Then U being $\mathcal{T}_1$ open $\mathcal{T}_1\text{cl}(U_x)$ is also disjoint from U or $\mathcal{T}_1\text{cl}(U_x) \cap U = \emptyset$ which is countable. Thus $x \notin (\mathcal{R}_1\text{-cl}\{U\})$.

Theorem 3.13. *If a BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is $(\mathcal{T}_1, \mathcal{T}_1)$ paracompact with respect to $\mathcal{T}_2$ and is pw $\mathcal{T}_2$ then $\mathcal{T}_1 \subseteq \mathcal{T}_2$.*

Proof: Let $U \in \mathcal{T}_1$ and let $x \in U$ then for every $y \notin U$ there always exists a $U_y \in \mathcal{T}_1$ containing y such that $x \notin \mathcal{T}_2\text{-cl}(U_y)$. Then $\mathcal{U} = \{U_y | y \notin U\} \cup \{U\}$ serves as a $\mathcal{T}_1$ open cover of X and therefore has a $\mathcal{T}_1$ open refinement say $\mathcal{V}$ that is $\mathcal{T}_2$ locally finite such that $\mathcal{V} = \{V_y | y \notin U\} \cup \{V\}$ where each V_y is contained in some U_y and V is contained in U . It is always possible to find such a precise refinement. Then, $x \in V$ since $x \notin U_y$ and hence $x \notin V_y$. Further $x \notin \mathcal{T}_2\text{-cl}(V_y)$ as $x \notin \mathcal{T}_2\text{-cl}(U_y)$. Finally, $\mathcal{V}$ being $\mathcal{T}_2$ locally finite, the arbitrary union of $\mathcal{T}_2$ closure of elements of $\mathcal{V}$ is $\mathcal{T}_2$ closed. Let $W = \cup\{V_y | y \notin U\}$. Then W is $\mathcal{T}_1$ open and $\mathcal{T}_2\text{-cl}W = \mathcal{T}_2\text{-cl}(V_y) = U(\mathcal{T}_2\text{-cl}\{V_y\})$. This implies that $X - U$ is contained in $\mathcal{T}_2\text{-cl}(W)$ or $X - \mathcal{T}_2\text{-cl}(W)$ is contained in U . Consequently, U is $\mathcal{T}_2$ open and hence $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Corollary 3.14. *A BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ which is both pw paracompact as well as pw $\mathcal{T}_2$ has identical topologies i.e. $\mathcal{T}_1 = \mathcal{T}_2$.*

Theorem 3.15. *A BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ which is $(\mathcal{T}_1, \mathcal{T}_1)$ paracompact with respect to $\mathcal{R}_1$ and also pw $\mathcal{T}_2$ exhibits the property that $\mathcal{T}_1 \subseteq \mathcal{R}_1$.*

Proof: Let $U \in \mathcal{T}_1$ and let $x \in U$ then for every $y \notin U$ there always exists a $U_y \in \mathcal{T}_1$ containing y such that $x \notin \mathcal{T}_2\text{-cl}(U_y)$. By Theorem 3.12, $\mathcal{T}_2\text{-cl}(U_y)$ contains $\mathcal{R}_1\text{-cl}(U_y)$ and therefore $x \notin \mathcal{R}_1\text{-cl}(U_y)$. Then $\mathcal{U} = \{U_y | y \notin U\} \cup \{U\}$ serves as a $\mathcal{T}_1$ open cover of X and therefore has a $\mathcal{T}_1$ open refinement say $\mathcal{V}$ that is $\mathcal{R}_1$ locally finite with the property that $\mathcal{V} = \{V_y | y \notin U\} \cup \{V\}$ where each V_y is contained in some U_y and U contains V . It is always possible to find such a precise refinement as before. Then, $x \in V$ since $x \notin U_y$ and hence $x \notin V_y$. Further, $x \notin \mathcal{R}_1\text{-cl}(V_y)$ as $x \notin \mathcal{R}_1\text{-cl}(U_y)$. Finally, $\mathcal{V}$ being $\mathcal{R}_1$ locally finite arbitrary union of $\mathcal{R}_1$ closure of elements of $\mathcal{V}$ is $\mathcal{R}_1$ closed. Let $W = \cup\{V_y | y \notin U\}$. Then W is $\mathcal{T}_1$ open and $\mathcal{R}_1\text{-cl}(W) = \mathcal{R}_1\text{-cl}(\cup\{V_y\}) = \cup(\mathcal{R}_1\text{-cl}\{V_y\})$. This implies that $X - U$ is contained in $\mathcal{R}_1\text{-cl}(W)$ or $X - \mathcal{R}_1\text{-cl}(W)$ is contained in U . Consequently, U is $\mathcal{R}_1$ open and hence $\mathcal{T}_1 \subseteq \mathcal{R}_1$.

Corollary 3.16. *A BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ which is pw Ω -paracompact and pw $\mathcal{T}_3$ has the property that $\mathcal{T}_1 \cup \mathcal{T}_2 \subseteq \mathcal{R}_1 \cap \mathcal{R}_2$.*

Definition 3.17. *A BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pw antilocally countable space if every nonempty member of $\mathcal{T}_1 \cup \mathcal{T}_2$ is uncountable.*

Theorem 3.18. *In a BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ which is pw antilocally countable, for any $\mathcal{T}_1$ -open set U , $\mathcal{T}_1\text{-cl}(U) \subseteq \mathcal{R}_2\text{-cl}(U)$*

Proof: Let $x \in \mathcal{T}_1\text{-cl}(U)$ and let V be an $\mathcal{R}_2$ -open neighbourhood of x . Then $\exists$ a $\mathcal{T}_1$ neighborhood U_x of x in X such that $\mathcal{T}_2\text{-cl}(U_x) - V = C$ where C is countable and does not contain x

$$\begin{aligned} &\Rightarrow \mathcal{T}_2\text{-cl}(U_x) - C \subseteq V \\ &\Rightarrow U_x - C \subseteq V \\ &\Rightarrow (U_x \cap U) - C \subseteq V \cap U \\ &\Rightarrow U_x \cap U \text{ is uncountable } X \text{ being pw antilocally countable} \\ &\Rightarrow V \cap U \text{ is not empty leading to the fact that } \mathcal{T}_2\text{-cl}(U) \subseteq \mathcal{R}_1\text{-cl}(U). \end{aligned}$$

Corollary 3.19. *In a BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ which is pw antilocally countable, if U is a $\mathcal{T}_1$ -open subset of X then $\mathcal{T}_1\text{-cl}(U) = \mathcal{R}_2\text{-cl}(U)$.*

Theorem 3.20. *In a BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ which is both pw p -space and pw $\mathcal{T}_1$, a collection of subsets of X is $\mathcal{T}_j$ locally finite if it is $\mathcal{R}_i$ -locally finite $\forall i, j \in \{1, 2\}, i \neq j$.*

Proof: Let $\mathcal{U} = \{U_\alpha | \alpha \in \Lambda\}$ be an $\mathcal{R}_1$ -locally finite collection of subsets of X . Consider an element $x \in X$ then, there always exists $U_x \in \mathcal{R}_1$ containing x with the property that $U_x \cap U_\alpha = \emptyset$ except for finitely many values of α . Now $U_x \in \mathcal{R}_1 \Rightarrow \exists$ a $\mathcal{T}_2$ open neighborhood V_x of x such that $\mathcal{T}_1\text{-cl}(V_x) - C \subseteq U_x$ or $V_x \cap (X - C) \subseteq U_x$. Now X is pw $\mathcal{T}_1$ and pw p -space therefore every singleton is $\mathcal{T}_2$ closed and consequently $(X - C)$ is $\mathcal{T}_2$ open being complement of countable collection of $\mathcal{T}_2$ closed sets. Thus U_x is a $\mathcal{T}_2$ open set intersecting finitely many members of $\mathcal{U}$.

Corollary 3.21. *If a BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pw $\mathcal{T}_1$ and pw p -space then it is pw paracompact if it is pw Ω -paracompact.*

Corollary 3.22. *A BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ which is both pw $\mathcal{T}_3$ as well as pw p -space will be pw paracompact if and only if it is pw Ω -paracompact.*

Theorem 3.23. *If a pw Ω -paracompact BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pw $\mathcal{T}_2$ and pw p -space then it is pw regular space.*

Proof: Consider an element $x \in X$ and consider a $\mathcal{T}_2$ closed set $F \subseteq X$ not containing x . Then for any $y \in F$ there exist disjoint sets $U_y \in \mathcal{T}_1$ containing x and $V_y \in \mathcal{T}_2$ containing y . Then the collection $\mathcal{U} = \{V_y | y \in F\} \cup \{X - F\}$ forms a $\mathcal{T}_2$ open cover of X and due to pw paracompactness of X has a $\mathcal{T}_2$ open $\mathcal{R}_2$ locally finite refinement $\mathcal{V}$. Let $W = \bigcup \{V \in \mathcal{V} | V \cap F \neq \emptyset\}$ then W is $\mathcal{T}_2$ open and contains F . Further $\mathcal{V}$ is $\mathcal{R}_2$ locally finite therefore we can assume that x has an $\mathcal{R}_2$ open neighbourhood V_x disjoint from F and intersecting with say $V_{y_1}, V_{y_2}, \dots, V_{y_n}$. Now each $V_{y_n} \in \mathcal{T}_2$ of $\mathcal{V}$ and corresponding to each $V_{y_n} \exists$ a $U_{y_n} \in \mathcal{T}_1$ containing x . Consider $U = V_x \cap U_{y_1} \cap U_{y_2} \cap \dots \cap U_{y_n}$. Then by Theorem 3.20, U is $\mathcal{T}_1$ open, contains x and is disjoint from W . Hence the result.

Theorem 3.24. *In a BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ which is pw Ω -paracompact, each of its $\mathcal{T}_i$ open cover has a refinement that can be decomposed into at most countably many families of $\mathcal{T}_i$ open sets which are $\mathcal{R}_i$ locally finite.*

Proof: Let $\mathcal{U}$ be the open cover and let $\mathcal{V}$ be the $\mathcal{T}_i$ open $\mathcal{R}_i$ locally finite refinement. Then $\mathcal{V} = \bigcup \mathcal{V}_n$ where each $\mathcal{V}_n = \mathcal{V}$ for every n serves the purpose.

Theorem 3.25. *If BTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ is such that each of its $\mathcal{T}_i$ open cover has a refinement that can be decomposed into at most countably many families of $\mathcal{T}_i$ open sets which are $\mathcal{R}_i$ locally finite then each of its $\mathcal{T}_i$ open cover has a refinement that is $\mathcal{R}_i$ locally finite.*

Proof: Take $\mathcal{U}$ as a $\mathcal{T}_1$ open cover and take $\mathcal{V} = \bigcup \mathcal{V}_n$ be the $\mathcal{T}_1$ open $\mathcal{R}_1$ locally finite refinement. Let $\mathcal{V}_n = \{V_\alpha | \alpha \in \Delta\}$ and let $W_n = \bigcup_{\alpha \in \Delta} V_\alpha | n \in \mathbb{N}$. Then $\{W_n | n \in \mathbb{N}\}$ is a cover of X . Let $X_n = W_n - \bigcup_{p < n} W_p$ and finally let us consider the collection $\{V_\alpha \cap X_n | \alpha \in \Delta, n \in \mathbb{N}\}$ then if n is the smallest integer such that x is contained in W_n then x is in X_n . Thus $\{X_n | n \in \mathbb{N}\}$ covers X , refines $\{W_n | n \in \mathbb{N}\}$ and consequently $\mathcal{U}$ and therefore is $\mathcal{R}_1$ locally finite.

Theorem 3.26. *In a pw regular BTS a $\mathcal{T}_i$ open cover having an $\mathcal{R}_i$ locally finite refinement has a $\mathcal{T}_j$ closed $\mathcal{R}_i$ locally finite refinement.*

Proof: Let $\mathcal{U}$ be a $\mathcal{T}_i$ open cover then $\forall x \in X \exists$ some U_x in $\mathcal{U}$ containing x . Since X possesses the property of pw regularity, $\exists$ a $\mathcal{T}_i$ open set say V_x with the property that $x \in V_x \subseteq \mathcal{T}_j \text{ cl}(V_x) \subseteq U_x$. Then $\{V_x | \mathcal{T}_j \text{ cl}(V_x) \subseteq U_x\}$ is a $\mathcal{T}_i$ open refinement of $\mathcal{U}$. Since this collection is a $\mathcal{T}_i$ open cover of X , it has an $\mathcal{R}_i$ locally finite refinement say $\mathcal{W}$ whose elements are contained in some V_x . It is easy to see that the collection of $\mathcal{T}_j$ closures of elements of $\mathcal{W}$ is the refinement of $\mathcal{U}$ and serves the purpose.

REFERENCES

- [1] A. C. Necati, *On Γ -Paracompact Spaces*, Konuralp Journal of Mathematics, 11 (2023), 77-81.
- [2] A. K. Banarjee, and R. Mondal, *Paracompactness in Bispace*, arXiv preprint arXiv: (2020), 2004.01209.
- [3] C. K. Goel and P. Agarwal, *A New Class of Open Sets*, International Research Journal of Mathematics, Engineering and Information Technology, 3 (2016), 1-9.
- [4] F. A. Abushaheen and H. Z. Hdeib, *p - $[a, b]$ -Paracompactness in Bitopological Spaces*, Italian Journal of Pure and Applied Mathematics, 42 (2019), 242250.
- [5] H. Singh, *On Pairwise Basis for Bitopological Space*, Research Review International Journal of Multidisciplinary, 3 (2018), 367-368.
- [6] I. L. Reilly, *On Bitopological Separation Properties*, Nanta Mathematica, 5 (1972), 14-25.
- [7] J. C. Kelly, *Bitopological Spaces*, Proceedings of London Mathematical Society, 13 (1963), 71-89.
- [8] J. Sanabria, E. Rosas, and C. Blanco, *Properties of Nearly S -Paracompact Spaces*, Proyecciones Journal of Mathematics, 42 (2023), 91-103.
- [9] M. Benbettiche, and H. Z. Hdeib, *Pairwise A -Paracompact Subsets*, Advances in Mathematics: Scientific Journal, 10(5) (2021), 26312639.
- [10] M. C. Dutta, *Paracompactness in Bitopological Spaces and an Application to Quasi-metric Spaces*, Indian Journal of Pure and Applied Mathematics, 8 (1977), 685690.
- [11] M. Ganster and I. L. Reilly, *On Pairwise Paracompactness*, J. Australian Math. Soc. (Series A), 53 (1992), 281-285.
- [12] M. K. Bose, A. R. Choudhary, and A. Mukharjee, *On Bitopological Paracompactness*, Matematicki Vesnik, 60 (2008), 255-259.
- [13] M. M. Kover, *A note on Raghavan-Reillys Pairwise Paracompactness*, International Journal of Mathematics and Mathematical Sciences, 24 (2000), 139-143.
- [14] N. L. Kalantan, *L -paracompactness and L_2 -paracompactness*, Hacettepe Journal of Mathematics and Statistics, 48 (2019), 779-784.
- [15] P. Agarwal and C. K. Goel, *Accretion of A New Bitopology*. International Journal of Recent Technology and Engineering 8(2) (2019), 3103-3106.
- [16] P. Agarwal and C. K. Goel, *Delineation of Ω -Bitopological Space*. Proceedings of Jangjeon Mathematical Society 22(4) (2019), 607-616.
- [17] P. Fletcher, H. B. Hoyle, and C. W. Patty, *The Comparison of Topologies*, Duke Math. J., 36 (1969), 325-331.
- [18] T. G. Raghavan and I. L. Reilly, *A New Bitopological Paracompactness*, Journal of the Australian Mathematical Society (Series A), 41 (1986), 268-274.

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